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LETTERS TO THE EDITOR

ON THE VIBRATION OF SATURATED LAYERED HALF-SPACE DUE TO LOW FREQUENCY EXCITATION

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1. INTRODUCTION

Since Biot's pioneering work on wave propagation in fluid-saturated porous solids [1], there have been numerous studies on various aspects of such a subject. Among them analytical study of boundary value (source) problems associated with a saturated half-space has increasingly attracted attention [2, 3]. Recently, an interesting attempt was made by Philippacopoulos [4] to analyze the response of a partially saturated layered half-space due to surface loads based on his previous work [2, 5]. Similar to previous studies, the Helmholtz decomposition suggested by Deresiewicz [6] was used to resolve the displacement vectors into potential functions and the classical formulation established by Biot was introduced. The integral solutions for surface displacements were derived, but numerical evaluations of the inverse Hankel transforms were not obtained since the solutions were complicated and intractable.

A similar problem is considered in this note for the purpose of applications in geomechanics and earthquake engineering. The present work differs from the previous study in that (1) a simple but useful, so called *u–p* formulation is adopted here to describe the dynamic behavior of saturated soils, where the terms relating to the fluid acceleration are neglected and the number of variables is effectively reduced [7, 8]. This approximation has been examined to be reasonable for most problems of earthquake analysis and soil dynamics (frequencies lower than this) and to be particularly economical and convenient in numerical analysis. Moreover, all the material properties are clearly defined and can be easily obtained from conventional soil tests. Therefore, the formulation is advantageous in engineering applications compared to the complete one of Biot's and has been widely used in geomechanics. (2) A simple and direct analysis approach is presented, which does not involve the complicated Helmholtz decomposition in poroelasticity. In references [4] and [5], however, altogether six potential functions were used to formulate the boundary problem. Analytic solutions are derived by Hankel transforms and verified by reducing to classical solutions of Lamb's problem. The evaluation of surface displacements in the space domain is given by using numerical integration. It should be mentioned that the present analysis formulation may also be applied to solve the problems corresponding to other loading cases (e.g., vertical ring loads at the surface or at a finite depth) and thereafter a set of fundamental solutions may be obtained. Further, it may be employed in conjunction with the initial parameter method [9] to conveniently derive the fundamental solutions for the quasi-static and dynamic responses of a multi-layered saturated soil deposit.

2. GOVERNING EQUATIONS AND THEIR SOLUTIONS

The system considered here is symmetric with respect to the z-axis, the plane $z = -h$ defines the free surface at which a point load is acting (Figure 1). Next, one considers the vibrations of the elastic dry soil layer and saturated half-space.

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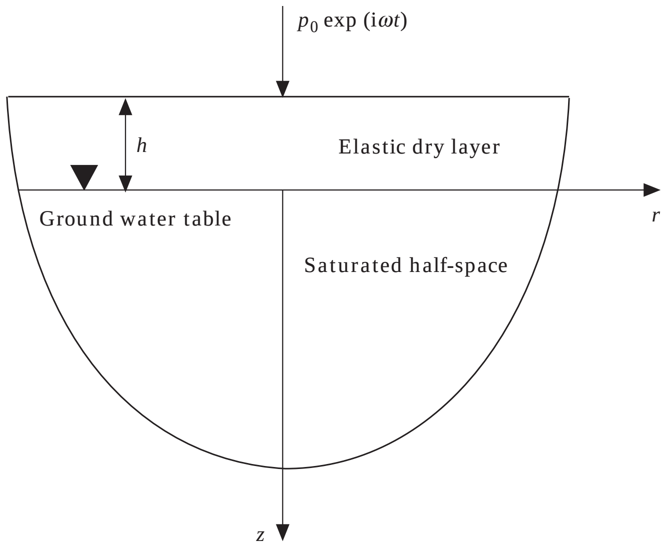

Figure 1. Saturated layer half-space.

2.1. *Vibration of elastic dry layer*

For the axisymmetric problem, the governing equations can be written as follows for a harmonic motion with a time-dependence exp(iωt) (e.g., $u_r = \bar{u}_r\, e^{i\omega t}$)

$$(\nabla^2 - 1/r^2)\bar{u}_r + [(\lambda' + \mu')/\mu']\, \partial\bar{e}/\partial r + (\rho'/\mu')\omega^2\bar{u}_r = 0, \tag{1}$$

$$\nabla^2\bar{u}_z + [(\lambda' + \mu')/\mu']\, \partial\bar{e}/\partial z + (\rho'/\mu')\omega^2\bar{u}_z = 0, \tag{2}$$

where λ', μ' are Lamé constants of the elastic dry layer; ρ' is mass density; e is volumetric strain; u_r, u_z are radial and vertical displacements, respectively; and ∇^2 denotes the Laplacian operator.

By taking $(\partial/\partial r)(1) + (1/r)(1) + (\partial/\partial z)(2)$, one can get

$$D'\nabla^2\bar{e} + \rho'\omega^2\bar{e} = 0, \tag{3}$$

where $D' = \lambda' + 2\mu'$. Applying zeroth order Hankel transforms [10] of equations (2) and (3), respectively, one can obtain

$$\mathrm{d}^2\tilde{u}_z/\mathrm{d}z^2 - q^2\tilde{u}_z + [(\lambda' + \mu')/\mu'](\mathrm{d}\tilde{e}/\mathrm{d}z) = 0, \qquad \mathrm{d}^2\tilde{e}/\mathrm{d}z^2 - p^2\tilde{e} = 0, \tag{4, 5}$$

where

$$p^2 = \xi^2 - \rho'\omega^2/D', \qquad q^2 = \xi^2 - \rho'\omega^2/\mu'$$

and the symbols " $\sim$ " on the top of the quantities denote Hankel transforms, ξ is the transform parameter.

Equations (4) and (5) can be solved as follows:

$$\tilde{e} = A_1 \cosh(pz) + B_1 \sinh(pz), \tag{6}$$

$$\tilde{u}_z = A_2 \sinh(qz) + B_2 \cosh(qz) - (D'p/\rho'\omega^2)[A_1 \sinh(pz) + B_1 \cosh(pz)], \tag{7}$$

where A_1, B_1, A_2, B_2 are integral constants.

With the definition of volumetric strain and (6) and (7), it is easy to give $\tilde{u}_r$ as

$$\xi\tilde{u}_r = (D'\xi^2/\rho'\omega^2)[A_1 \cosh(pz) + B_1 \sinh(pz)] - [A_2 q \cosh(qz) + B_2 q \sinh(qz)]. \tag{8}$$

From the familiar stress–strain relations and after some substitution, one obtains the stresses

$$\frac{\tilde{\tau}_{rz}}{2\mu'} = \frac{D'p\xi}{\rho'\omega^2}\,[A_1 \sinh(pz) + B_1 \cosh(pz)] - \frac{q^2+\xi^2}{2\xi}\,[A_2 \sinh(qz) + B_2 \cosh(qz)], \quad (9)$$

$$\frac{\tilde{\sigma}_z}{2\mu'} = \left(\frac{D'}{2\mu'} - \frac{D'^2\xi^2}{\rho'\omega^2}\right)[A_1 \cosh(pz) + B_1 \sinh(pz)] + q[A_2 \cosh(qz) + B_2 \sinh(qz)]. \quad (10)$$

Finally, equations (7–10) can be conveniently rewritten as the following matrix form

$$\{\mathbf{V}\} = [\boldsymbol{\phi}]\{\mathbf{W}\}, \quad (11)$$

in which

$$\{\mathbf{V}\} = [\tilde{u}_r \;\; \tilde{u}_z \;\; \tilde{\tau}_{rz} \;\; \tilde{\sigma}_z]^{\mathrm{T}}, \qquad \{\mathbf{W}\} = [A_1 \;\; B_1 \;\; A_2 \;\; B_2]^{\mathrm{T}}.$$

The elements of matrix $[\boldsymbol{\phi}]$ are listed in the Appendix.

2.2. *Vibration of saturated half-space*

As stated previously, for most problems in earthquake analysis and soil dynamics, the terms involving the fluid acceleration in Biot's equations can be neglected with confidence. The governing equations can thus be written as follows for the axisymmetric problem without body forces (see reference [7], equations 5, 7 and 10):

$$\mu(\nabla^2 - 1/r^2)u_r + (\lambda + \mu)\,\partial e/\partial r - \partial p_f/\partial r = \rho\,\partial^2 u_r/\partial t^2, \quad (12)$$

$$\mu\nabla^2 u_z + (\lambda + \mu)\,\partial e/\partial z - \partial p_f/\partial z = \rho\,\partial^2 u_z/\partial t^2, \quad (13)$$

$$-\partial p_f/\partial r = (\rho_f g/k)\,\partial w_r/\partial t + \rho_f\,\partial^2 u_r/\partial t^2, \qquad -\partial p_f/\partial z = (\rho_f g/k)\,\partial w_z/\partial t + \rho_f\,\partial^2 u_z/\partial t^2, \quad (14, 15)$$

where λ and μ are Lamé constants, u_r, u_z and w_r, w_z are solid and fluid phase displacements, respectively; e is volumetric strain of solid phase; p_f is pore pressure, ρ and ρ_f are mass densities of the total unit and the pore fluid, respectively, $\rho = (1-n)\rho_s + n\rho_f$, n is porosity, ρ_s is the density of soil grains, k is the coefficient of permeability with unit of (m/s).

For incompressible pore fluid, which is a generally acceptable notation in soil mechanics [8, 11], the flow continuity is given as (the upper dots denote the derivation with respect to time)

$$\partial\dot{w}_r/\partial r + \dot{w}_r/r + \partial\dot{w}_z/\partial z + \partial\dot{u}_r/\partial r + \dot{u}_r/r + \partial\dot{u}_z/\partial z = 0. \quad (16)$$

The constitutive laws and the effective stress principal (tension positive) for saturated soils can be written as

$$\sigma_z' = \lambda e + 2\mu\,\partial u_z/\partial z, \qquad \tau_{rz} = \mu(\partial u_r/\partial z + \partial u_z/\partial r), \qquad \sigma_z = \sigma_z' - p_f, \quad (17\text{–}19)$$

where σ_z' and σ_z are the effective and total stresses in soil mechanics [7], respectively.

Conventionally, the governing equations are solved by introducing displacement decomposition based on Helmholtz representation, in which four potentials are employed, two are associated with the solid phase, while the other two are associated with the flow of the pore fluid relative to the solid [4–6]. Next, we proceed to solve the equations without the aid of potential theory.

By taking $(\partial/\partial r)(14) + (1/r)(14) + (\partial/\partial z)(15)$ and with equation (16), one has

$$\nabla^2 p_f = (1/k')\dot{e} - \rho_f\ddot{e}, \quad (20)$$

where $k' = k/\rho_f g$.

Similarly, taking $(\partial/\partial r)(12) + (1/r)(12) + (\partial/\partial z)(13)$ and with the above equation gives

$$\nabla^2 e = (1/k'D)\dot{e} + [(\rho - \rho_f)/D]\ddot{e}. \tag{21}$$

where $D = \lambda + 2\mu$. Now the governing equations can be converted into four alternate uncoupled equations (12), (13), (20) and (21). It should be noted that only three of these four equations are independent. Similarly, with the assumption that the motion is time-harmonic, and applying the first order Hankel transform to equation (12) and zeroth order Hankel transforms to equations (13), (20) and (21), respectively, one has

$$\mathrm{d}^2\tilde{u}_r/\mathrm{d}z^2 - \delta^2\tilde{u}_r = [(\lambda + \mu)/\mu]\xi\tilde{e} - (1/\mu)\xi\tilde{p}_f, \tag{22}$$

$$\mathrm{d}^2\tilde{u}_z/\mathrm{d}z^2 - \delta^2\tilde{u}_z = -[(\lambda + \mu)/\mu]\,\mathrm{d}\tilde{e}/\mathrm{d}z + (1/\mu)\,\mathrm{d}\tilde{p}_f/\mathrm{d}z, \tag{23}$$

$$\mathrm{d}^2\tilde{p}_f/\mathrm{d}z^2 - \xi^2\tilde{p}_f = \beta^2\tilde{e}, \qquad \mathrm{d}^2\tilde{e}/\mathrm{d}z^2 - \alpha^2\tilde{e} = 0, \tag{24, 25}$$

where

$$\alpha^2 = \xi^2 - \eta^2, \qquad \beta^2 = \rho\omega^2 - D\eta^2, \qquad \eta^2 = \frac{\rho - \rho_f}{D}\omega^2 - \frac{\mathrm{i}\omega}{k'D}, \qquad \delta^2 = \xi^2 - \frac{\rho\omega^2}{\mu}. \tag{26}$$

It is seen that these equations can be easily solved one by one. First, under the wave radiation condition, only outgoing waves are allowed in the half-space. Thus, with the assumptions of $\mathrm{Re}(\alpha) < 0$ and $\mathrm{Re}(\delta) < 0$, equation (25) can be solved for $\tilde{e}$ and then given $\tilde{e}$, equation (24) can be solved for $\tilde{p}_f$. Similarly, the expressions for $\tilde{u}_z$, $\tilde{u}_r$ can be obtained from equations (23) and (22). Finally the solutions are given as follows

$$\tilde{e} = A_1'\,\mathrm{e}^{\alpha z}, \qquad \tilde{p}_f = -(\beta^2/\eta^2)A_1'\,\mathrm{e}^{\alpha z} + A_2'\mu\,\mathrm{e}^{-\xi z}, \tag{27, 28}$$

$$\tilde{u}_z = -(\alpha/\eta^2)A_1'\,\mathrm{e}^{\alpha z} - (\mu\xi/\rho\omega^2)A_2'\,\mathrm{e}^{-\xi z} + (1/\xi)A_3'\,\mathrm{e}^{\delta z}, \tag{29}$$

$$\tilde{u}_r = (\xi/\eta^2)A_1'\,\mathrm{e}^{\alpha z} - (\mu\xi/\rho\omega^2)A_2'\,\mathrm{e}^{-\xi z} + (1/\delta)A_4'\,\mathrm{e}^{\delta z}. \tag{30}$$

Among the four integral constants in the above equations, only three are independent. With the relations of (27), (29) and (30), one obtains

$$A_4' = -(\delta/\xi)A_3'. \tag{31}$$

Similarly, based on the stress–strain relations (17)–(19), the stresses can be given as

$$\frac{\tilde{\tau}_{rz}}{2\mu} = \frac{\alpha\xi}{\eta^2}A_1'\,\mathrm{e}^{\alpha z} + \frac{\mu\xi^2}{\rho\omega^2}A_2'\,\mathrm{e}^{-\xi z} - \tfrac{1}{2}\left(\frac{\xi^2 + \delta^2}{\delta\xi}\right)A_3'\,\mathrm{e}^{\delta z}, \tag{32}$$

$$\frac{\tilde{\sigma}_z}{2\mu} = \left(\frac{\lambda}{2\mu} - \frac{\alpha^2}{\eta^2} + \frac{\beta^2}{2\mu\eta^2}\right)A_1'\,\mathrm{e}^{\alpha z} + \left(\frac{\mu\xi^2}{\rho\omega^2} - \tfrac{1}{2}\right)A_2'\,\mathrm{e}^{-\xi z}. \tag{33}$$

One can now arrange equations (29), (30), (32) and (33) in matrix form as

$$\{\mathbf{Y}\} = [\boldsymbol{\varphi}]\{\mathbf{X}\}, \tag{34}$$

where

$$\{\mathbf{Y}\} = [\tilde{u}_r\ \ \tilde{u}_z\ \ \tilde{\tau}_{rz}\ \ \tilde{\sigma}_z]^{\mathrm{T}}, \qquad \{\mathbf{X}\} = [A_1'\ \ A_2'\ \ A_3']^{\mathrm{T}}.$$

The elements of matrix $[\boldsymbol{\varphi}]$ are given in the Appendix.

3. BOUNDARY VALUE PROBLEM

First, one considers the hydraulic interface condition. The interface is usually assumed to be free draining, that is the pore pressure is zero. With this condition, the integral constant A_2' in equation (34) can be eliminated and can thus be simplified as

$$\{\mathbf{Y}\} = [\boldsymbol{\varphi}]_d\{\mathbf{X}\}, \tag{35}$$

where

$$\{\mathbf{X}\} = [A_1' \;\; A_3']^{\mathrm{T}}.$$

The elements of matrix $[\boldsymbol{\varphi}]_d$ are listed in the Appendix.

Placing $z = -h$, $z = 0^-$ over equation (11) and placing $z = 0^+$ over equation (35), and introducing the following continuity conditions at the interface

$$\begin{Bmatrix} u_r \\ u_z \\ \tau_{rz} \\ \sigma_z \end{Bmatrix}_{z=0^-} = \begin{Bmatrix} u_r \\ u_z \\ \tau_{rz} \\ \sigma_z \end{Bmatrix}_{z=0^+}, \tag{36}$$

one has

$$\{\mathbf{V}\}_{z=-h} = [\mathbf{Q}]\{\mathbf{X}\}, \tag{37}$$

where

$$[\mathbf{Q}] = [\boldsymbol{\phi}]_{z=-h}[\boldsymbol{\phi}]^{-1}_{z=0}[\boldsymbol{\varphi}]_{z=0}. \tag{38}$$

The two integral constants in equation (37) can be determined from the following boundary conditions at $z = -h$ ($\delta(r)$ is Dirac delta function)

$$\bar{\tau}_{rz} = 0, \qquad \bar{\sigma}_z = -p_0\delta(r)/2\pi r. \tag{39, 40}$$

Then, the surface radial and vertical displacements can be further achieved as follows:

$$\begin{Bmatrix} \tilde{u}_r \\ \tilde{u}_z \end{Bmatrix}_{z=-h} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}\begin{bmatrix} Q_{31} & Q_{32} \\ Q_{41} & Q_{42} \end{bmatrix}^{-1}\begin{Bmatrix} 0 \\ -p_0/2\pi \end{Bmatrix}. \tag{41}$$

Applying the inverse Hankel transforms of the above two equations, one obtains the solutions in terms of integration:

$$\bar{u}_r = \frac{p_0}{2\pi}\int_0^\infty \xi\,\frac{Q_{11}Q_{32} - Q_{12}Q_{31}}{Q_{31}Q_{42} - Q_{41}Q_{32}}\,J_1(\xi r)\,\mathrm{d}\xi, \tag{42}$$

$$\bar{u}_z = \frac{p_0}{2\pi}\int_0^\infty \xi\,\frac{Q_{21}Q_{32} - Q_{22}Q_{31}}{Q_{31}Q_{42} - Q_{41}Q_{32}}\,J_0(\xi r)\,\mathrm{d}\xi. \tag{43}$$

After the surface displacements are given, the stresses and displacements in the dry layer and the saturated half-space can be conveniently achieved. In this note, one only focuses on the surface displacements.

4. VERIFICATION OF SOLUTIONS

One now proceeds to demonstrate the accuracy of the solutions by considering a limiting case. When the thickness of the overlying dry layer goes to zero ($h\to 0$), the model considered would become the saturated half-space model. Equation (37) becomes

$$\{\mathbf{V}\}_{z=-h} = [\boldsymbol{\varphi}]_{d,z=0}\{\mathbf{X}\}. \tag{44}$$

By introducing the boundary conditions, the solutions can be obtained as

$$\bar{u}_r = \frac{p_0}{2\pi}\int_0^\infty \xi^2 \frac{\Delta_1}{\Delta} J_1(\xi r)\,\mathrm{d}\xi, \qquad \bar{u}_z = \frac{p_0}{2\pi}\int_0^\infty \xi \frac{\Delta_2}{\Delta} J_0(\xi r)\,\mathrm{d}\xi, \tag{45, 46}$$

where

$$\Delta_1 = \mu(\rho\omega^2 - \beta^2)(\xi^2 + \delta^2) - 2\mu\delta(\alpha\rho\omega^2 + \beta^2\xi), \tag{47}$$

$$\Delta_2 = \mu(\alpha\rho\omega^2 + \beta^2\xi)(\xi^2 - \delta^2) - 4\mu\beta^2\xi^3, \tag{48}$$

$$\Delta = \mu\rho\omega^2(2\mu\alpha^2 - \lambda\eta^2)(\xi^2 + \delta^2) - \mu^2\beta^2\xi^2(\xi^2 + \delta^2) - 4\mu^2\xi^2\delta(\alpha\rho\omega^2 + \beta^2\xi). \tag{49}$$

When there is no pore fluid in the material ($\rho_f = 0$), the above model can be further simplified as a conventional half-space (one phase media). The parameters in equation (26) are reduced to

$$\eta^2 = \rho\omega^2/D, \qquad \beta^2 = 0, \qquad \alpha^2 = \xi^2 - \rho\omega^2/D \tag{50}$$

Then, Δ_1, Δ_2 and Δ can be simplified and the solutions for the surface displacements of the conventional half-space take the following forms:

$$\bar{u}_r = \frac{p_0}{2\pi\mu}\int_0^\infty \xi^2 \frac{\xi^2 + \delta^2 - 2\alpha\delta}{(\xi^2 + \delta^2)^2 - 4\alpha\delta\xi^2} J_1(\xi r)\,\mathrm{d}\xi, \tag{51}$$

$$\bar{u}_z = \frac{p_0}{2\pi\mu}\int_0^\infty \xi \frac{\alpha(\xi^2 - \delta^2)}{(\xi^2 + \delta^2)^2 - 4\alpha\delta\xi^2} J_0(\xi r)\,\mathrm{d}\xi. \tag{52}$$

In view of the assumptions of $\mathrm{Re}(\alpha) <$, $\mathrm{Re}(\delta) > 0$, it is found that equation (51) and (52) are the conventional solutions of Lamb's problem [12].

5. NUMERICAL EXAMPLE

A simple example is presented in this section to illustrate applications of the present theory. To obtain the surface displacements (42) and (43), numerical integration has to be employed. In the calculation, the physical properties of the dry soil layer are: $\mu' = 8{\cdot}6$ MPa; $v' = 0{\cdot}3$; $\rho' = 1800$ kg/m^3 where v' is Poisson's ratio. The properties of saturated soils are: $\mu = 6{\cdot}6$ MPa; $v = 0{\cdot}35$; $\rho_s = 2600$ kg/m^3; $\rho_f = 1000$ kg/m^3 $n = 0{\cdot}4$ $k = 10^{-5}$ m/s.

The exciting load is taken to be 100 N, with frequency of 10 Hz. Figure 2 shows the variations of surface vertical and radial displacements in the near field. Two cases for the ground water table are presented. In case 1, the water table is assumed to be located at the depth of 1·5 m and in the other case, it is located at 1·0 m.

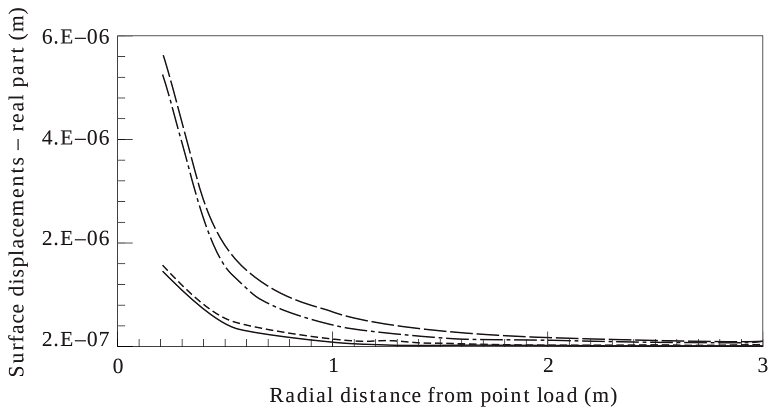

Figure 2. Surface vertical and radial displacements in the near field: — — —, vertical (h = 1·5 m); ----, radial (h = 1·5 m); — - —, vertical (h = 1·0 m); ——, radial (h = 1·0 m).

6. CONCLUSIONS

An analysis has been made of the response of a saturated layered half-space subjected to a harmonic surface loading based on the *u–p* dynamic formulation for two-phase material. A simple and direct approach is presented, which does not rely on the use of conventional Helmholtz representation in poroelasticity. Analytic solutions are obtained and verified and an example application is given. The analysis formulation presented in this note shows potential applications in solving other related boundary value problems associated with saturated uniform or layered soils in practical engineering.

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APPENDIX

The elements of matrix $[\boldsymbol{\phi}]$

$$\phi_{11} = \frac{D'\xi}{\rho'\omega^2}\cosh(pz), \qquad \phi_{12} = \frac{D'\xi}{\rho'\omega^2}\sinh(pz), \qquad \phi_{13} = -\frac{q}{\xi}\cosh(qz),$$

$$\phi_{14} = -\frac{q}{\xi}\sinh(qz),$$

$$\phi_{21} = -\frac{D'p}{\rho'\omega^2}\sinh(pz), \qquad \phi_{22} = -\frac{D'p}{\rho'\omega^2}\cosh(pz), \qquad \phi_{23} = \sinh(qz),$$

$$\phi_{24} = \cosh(qz),$$

$$\phi_{31} = \frac{2\mu'D'p\xi}{\rho'\omega^2}\sinh(pz), \qquad \phi_{32} = \frac{2\mu'D'p\xi}{\rho'\omega^2}\cosh(pz), \qquad \phi_{33} = -\frac{q^2+\xi^2}{\xi}\mu'\sinh(qz),$$

$$\phi_{34} = -\frac{q^2+\xi^2}{\xi}\mu'\cosh(qz), \qquad \phi_{41} = \left(D' - \frac{2\mu'D'\xi^2}{\rho'\omega^2}\right)\cosh(pz),$$

$$\phi_{42} = \left(D' - \frac{2\mu'D'\xi^2}{\rho'\omega^2}\right)\sinh(pz),$$

$$\phi_{43} = 2\mu'q\cosh(qz) \qquad \phi_{44} = 2\mu'q\sinh(qz).$$

The elements of matrix $[\boldsymbol{\varphi}]$ are as follows

$$\varphi_{11} = \frac{\xi}{\eta^2}e^{\alpha z}, \qquad \varphi_{12} = -\frac{\mu\xi}{\rho\omega^2}e^{-\xi z}, \qquad \varphi_{13} = -\frac{1}{\xi}e^{\delta z}, \qquad \varphi_{21} = -\frac{\alpha}{\eta^2}e^{\alpha z},$$

$$\varphi_{22} = -\frac{\mu\xi}{\rho\omega^2}e^{-\xi z}, \qquad \varphi_{23} = \frac{1}{\delta}e^{\delta z},$$

$$\varphi_{31} = \frac{2\mu\alpha\xi}{\eta^2}e^{\alpha z}, \qquad \varphi_{32} = \frac{2\mu^2\xi^2}{\rho\omega^2}e^{-\xi z}, \qquad \varphi_{33} = -\mu\frac{\xi^2+\delta^2}{\delta\xi}e^{\delta z},$$

$$\varphi_{41} = \left(\lambda - \frac{2\mu\alpha^2}{\eta^2} + \frac{\beta^2}{\eta^2}\right)e^{\alpha z} \qquad \varphi_{42} = \left(\frac{\mu\xi^2}{\rho\omega^2} - 1\right)\mu\, e^{-\xi z} \qquad \varphi_{43} = 2\mu\, e^{\delta z}.$$

The elements of matrix $[\boldsymbol{\varphi}]_d$

$$\varphi_{11} = \frac{\xi}{\eta^2}e^{\alpha z} - \frac{\beta^2\xi}{\eta^2\rho\omega^2}e^{-\xi z}, \qquad \varphi_{12} = -\frac{1}{\xi}e^{\delta z}, \qquad \varphi_{21} = -\frac{\alpha}{\eta^2}e^{\alpha z} - \frac{\beta^2\xi}{\eta^2\rho\omega^2}e^{-\xi z},$$

$$\varphi_{22} = \frac{1}{\delta}e^{\delta z}, \qquad \varphi_{31} = \frac{2\mu\alpha\xi}{\eta^2}e^{\alpha z} + \frac{2\mu\xi^2\beta^2}{\eta^2\rho\omega^2}e^{-\xi z}, \qquad \varphi_{32} = -\mu\left(\frac{\xi^2+\delta^2}{\xi\delta}\right)e^{\delta z},$$

$$\varphi_{41} = \left(\lambda - \frac{2\mu\alpha^2}{\eta^2} + \frac{\beta^2}{\eta^2}\right)e^{\alpha z} + \left(\frac{\mu\xi^2\beta^2}{\eta^2\rho\omega^2} - \frac{\beta^2}{\eta^2}\right)e^{-\xi z}, \qquad \varphi_{42} = 2\mu\, e^{\delta z}.$$